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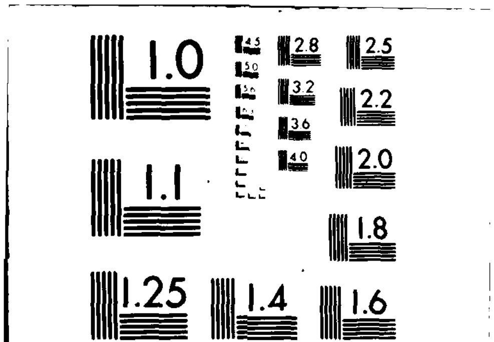

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QUANTILE FUNCTIONS, CONVERGENCE IN QUANTILE, AND EXTREME VALUE DISTRIBUTION THEORY

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Technical Report No. B-3

November 1980

Texas A & M Research Foundation

Project No. 4226

"Robust Statistical Data Analysis and Modeling"

Sponsored by the U.S. Army Research Office
Grant DAAG29-80-C-0070

Professor Emanuel Parzen, Principal Investigator

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: TR-B-3	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Quantile Functions, Convergence in Quantile, and Extreme Value Distribution Theory.		5. TYPE OF REPORT & PERIOD COVERED: Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Emanuel/Parzen		8. CONTRACT OR GRANT NUMBER(s): DAA29-80-C-0070
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Texas A&M University, Institute of Statistics, College Station, TX 77843		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Army Research Office		12. REPORT DATE: November 1980
		13. NUMBER OF PAGES: 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report): Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Quantile functions, density quantile function, tail exponents, convergence in distribution, converge in quantile, extreme value distributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

Unclassified

QUANTILE FUNCTIONS, CONVERGENCE IN QUANTILE, AND EXTREME VALUE DISTRIBUTION THEORY

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Abstract

The aim of this paper is to summarize the probability theory of quantile functions. The contributions of this paper are: (1) to emphasize the duality of quantile functions with distribution functions (Section 1); (2) to explicitly define the notions of "convergence in quantile" and "convergence in r-mean quantile" (Section 2); (3) provide simple proofs of the distribution theory of extreme values (Section 4); and (4) emphasize the role of tail exponents of quantile functions and density-quantile functions in providing easy to apply criteria for the extreme value distributions corresponding to a specified distribution.

AMS 1970 Subject Classification. Primary 60F05, 62E20.

Key Words: Quantile functions, density quantile function, tail exponents, convergence in distribution, converge in quantile, extreme value distributions.

[1] This research was supported in part by the Army Research Office (Grant DAA29-80-C-0070) .

1. Quantile Functions and Their Properties

Statisticians and probabilists have traditionally used distribution functions to study the properties of random variables. Quantile functions (the inverses of distribution functions) have in the past decade become increasingly used, but their elegant properties do not seem to be summarized in any reference.

Quantile functions are important for statistical data analysis; their use was pioneered by Tukey (1965) and Wilk and Gnanadesikan (1968). Quantile functions are advocated by Parzen (1979) as providing an approach to probability-based data analysis.

Quantile functions are important in probability theory for the study of invariance principles (see Major (1978). Quantile functions as a distance between probability measures were used by Mallows (1972) and is applied to the asymptotic theory of bootstrap methods by Bickel and Freedman (1981).

A general distribution function $F(x)$, $-\infty < x < \infty$, defined by $F(x) = \Pr[X \leq x]$, is characterized by the properties that it is (1) non-decreasing; (2) continuous from the right

$$F(x) = F(x+0) = \lim_{0<\varepsilon\to 0} F(x+\varepsilon) \;;$$

and (3) $F(-\infty) = 0$, $F(\infty) = 1$. Its quantile function, denoted $Q(u)$, $0 \leq u \leq 1$, or $F^{-1}(u)$, $0 \leq u \leq 1$, is defined by

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\} \,.$$

$Q(u)$ is characterized by the properties that it is (1) non-decreasing, and (2) continuous from the left.

An important example of a quantile function is the following: Let $F(x)$ be purely discrete, with jumps at $t_1 < t_2 < \ldots < t_k$ and values

$$F(t_j) = u_j \ , \quad j = 1, \ldots, k \ ,$$

where $0 < u_1 < \ldots < u_k = 1$. Then for $j = 1, 2, \ldots, k$ (defining $u_0 = 0$)

$$Q(u) = t_j \ , \quad u_{j-1} < u \leq u_j \ .$$

The distribution function $F(\cdot)$ of a random variable X is often assumed to be of the form

$$F(x) = F_0\left(\frac{x-\mu}{\sigma}\right) \ ,$$

where $F_0(\cdot)$ is a known standard distribution, and μ and σ are unknown parameters (to be estimated), called location and scale parameters respectively.

Theorem 1A. The quantile function $Q(u)$ corresponding to $F(x) = F_0\left(\frac{x-\mu}{\sigma}\right)$ is $Q(u) = \mu + \sigma\, Q_0(u)$, where $Q_0(u) = F_0^{-1}(u)$.

Proof: $F(x) = F_0\left(\frac{x-\mu}{\sigma}\right) \geq u$ iff $\frac{x-\mu}{\sigma} \geq Q_0(u)$ iff $x \geq \mu + \sigma Q_0(u)$. Therefore $\mu + \sigma Q_0(u)$ equals the inf of all x such that $F(x) \geq u$.

Quantile Functions of Standard Continuous Distributions. Some important distribution functions are:

Standard Normal. For $-\infty < x < \infty$

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy, \quad \phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-(\frac{1}{2})x^2} \ .$$

Standard Exponential. For $x > 0$

$$F_0(x) = 1-e^{-x} , \qquad f_0(x) = e^{-x} .$$

Standard Uniform. For $0 < x < 1$

$$F_0(x) = x , \qquad f_0(x) = 1 .$$

Standard Cauchy. For $-\infty < x < \infty$

$$F_0(x) = \frac{1}{\pi} \tan^{-1}x + \frac{1}{2} , \quad f_0(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

The corresponding quantile functions are

Standard Normal. $\Phi^{-1}(u)$, $0 \leq u \leq 1$

Standard Exponential. $Q_0(u) = F_0^{-1}(u) = \log (1-u)^{-1}$.

Standard Uniform. $Q_0(u) = u$.

Standard Cauchy. $Q_0(u) = \tan\pi (u-\frac{1}{2})$.

To illustrate how one computes a quantile function, consider the standard exponential; one writes $x = Q_0(u)$ satisfies $u = F_0(x) = 1-e^{-x}$, whence $1-u = e^{-x}$, $\log (1-u) = -x$.

We leave to the reader the proofs of the following basic properties of quantile functions.

Theorem 1B: Inverse Identities. For any x, x_1, x_2 in $-\infty < x < \infty$ and u, u_1, u_2 in $0 < u < 1$:

(1) $F(x) \geq u$ iff $Q(u) \leq x$;

(2) $F(x) < u$ iff $Q(u) > x$;

(3) $F(x_1) < u \leq F(x_2)$ iff $x_1 < Q(u_1) \leq x_2$;

(4) $FQ(u) \geq u$;

(5) $QF(x) \leq x$.

<u>Theorem 1C</u>: <u>Continuity Points of Q</u>:

(0) Q is continuous from the left; $Q(u) = \lim_{0<\varepsilon\to 0} Q(u-\varepsilon)$;

(1) $FQ(u) = u$ if $x = Q(u)$ is a continuity point of F ;

(2) $QF(x) = x$ if $u = F(x)$ is a continuity point of Q ;

(3) u is a continuity point of Q iff $F(Q(u) + \varepsilon) > u$ for all $\varepsilon > 0$;

(4) u is not a continuity point of Q iff $u = F(x) = F(x+\varepsilon)$ for some x and $\varepsilon > 0$ (in words, F(x) is constant over an interval) ;

(5) if F is continuous and strictly increasing, then every u in $0 < u < 1$ is a continuity point of Q ;

(6) if F is discrete, then the values of F(x) at the discontinuity points of F are the discontinuity points of Q ;

(7) there are at most a countable infinity of points u which are not continuity points of Q .

<u>Simulation and Representation</u>. To simulate on a computer a random sample $X_1, \ldots, X_n$ from a distribution F(x), one approach is to simulate $U_1, \ldots, U_n$ from a standard uniform distribution and form $X_1 = Q(U_1), \ldots, X_n = Q(U_n)$. The validity of this algorithm is a consequence of the Representation Theorem in which U denotes a standard uniform random variable.

<u>Definition</u>. Two random variables X and Y are said to be identically distributed, denoted $X \overset{D}{=} Y$, if for every x in $-\infty < x < \infty$

$$F_X(x) = \Pr(X \le x) = \Pr(Y \le x) = F_Y(x)$$

<u>Theorem 1D</u>: <u>Representation Identity</u>. $X \overset{D}{=} Q(U)$.

Proof: Since $[Q(U) \leq x]$ is equivalent to $[U \leq F(x)]$

$$\Pr[Q(U) \leq x] = \Pr[U \leq F(x)] = F(x).$$

Theorem 1E: Probability Integral Transformation. When $F(\cdot)$ is continuous, $F(X) \overset{D}{=} U$.

Proof: Since $[F(X) \geq u]$ is equivalent to $[X \geq Q(u)]$

$$\Pr[F(X) \geq u] = P[X \geq Q(u)] = 1-FQ(u) = 1-u.$$

The Representation Identity yields immediately a formula for the evaluation of expectations and moments.

Theorem 1F: Expectation Identity.

$$E[g(X)] = E[gQ(U)] = \int_0^1 g[Q(u)]du$$

The mean μ and variance σ^2 are given by

$$\mu = \int_0^1 Q(u)\ du, \ \sigma^2 = \int_0^1 [Q(u)-\mu]^2\ du \quad .$$

Another property of quantile functions is how they behave under monotone transformations of random variables.

Let $Y = g(X)$ where $g(x)$ is a non-decreasing function continuous from the left. Define

$$g^{-1}(y) = \sup\ \{x: g(x) \leq y\} \ .$$

Then $g(x) \leq y$ iff $x \leq g^{-1}(y)$. Consequently

$$F_Y(y) = \Pr\ [Y \leq y] = \Pr[g(X) \leq y] = \Pr[X \leq g^{-1}(y)]$$

$$= F_X(g^{-1}(y)) \ .$$

Therefore

$$F_Y(y) \geq u \text{ iff } F_X(g^{-1}(y)) \geq u \text{ iff } g^{-1}(y) \geq Q_X(u)$$

$$\text{iff } y \geq gQ_X(u) \ .$$

Therefore the smallest y such that $F_Y(y) \geq u$ equals $gQ_X(u)$, and the following theorem has been proved.

<u>Theorem</u> <u>1G</u>: <u>Transformation Identity</u>. Let g be a monotone (increasing or decreasing) function which is continuous from the left. The quantile function $Q_Y(u)$ of $Y = g(X)$ can be expressed in terms of the quantile function $Q_X(u)$ of X as follows:

$$Q_Y(u) = g(Q_X(u)) \text{ if } g \text{ increasing,}$$

If X is continuous, then

$$Q_Y(u) = g(Q_X(1-u)) \quad \text{if } g \text{ decreasing.}$$

To treat the case of g decreasing, it sufficies to treat the special case of $g(x) = -x$ (since $-g(x)$ is increasing if $g(x)$ is decreasing).

<u>Theorem</u> <u>1H</u>: If X is continuous

$$Q_{-X}(u) = -\,Q_X(1-u) \ .$$

<u>Proof</u>: We first write non-rigorously

$$u = F_{-X}(y) = 1-F_X(-y) \ , \quad -y = Q_X(1-u), \quad Q_Y(u) = -Q_X(1-u).$$

To be more rigorous, we write

$$\inf\{y: F_{-X}(y) \geq u\} = \inf\{y: 1-u \geq F_X(-y)\}$$

$$= -\sup\{z: 1-u \geq F_X(z)\} = -Q_X(1-u) \ .$$

<u>Theorem</u> <u>1I</u>: <u>Applications of the Transformation Identity</u>:

$$Y = \mu + \sigma X \ , \qquad Q_Y(u) = \mu + \sigma Q_X(u);$$

$$Y = -\log X, \qquad Q_Y(u) = -\log Q_X(1-u);$$

$$Y = 1/X \ , \qquad Q_Y(u) = 1/Q_X(1-u) \ .$$

<u>Theorem</u> <u>1J</u>: <u>Converse Transformation Identity</u>. If X is continuous, and g is increasing, and

$$\Pr[Y \leq \mu + \sigma g(x)] = F_X(x) \ , \quad -\infty < x < \infty \ ,$$

then $Q_Y(u) = \mu + \sigma g(Q_X(u))$ and $Y \overset{D}{=} \mu + \sigma g(X)$

<u>Proof</u>: Let $x = Q_X(u)$. Then $\Pr[Y \leq \mu+\sigma g(Q_X(u))] = F_X Q_X(u) = u$.

2. Convergence in Quantile

Definition: Convergence in Distribution. A sequence of random variables X_n, with distribution functions $F_n(x)$, are said to converge in distribution to X with distribution function F(x), if

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ at all continuity points } x \text{ of } F(\cdot).$$

We then write

$$\lim_{n\to\infty} X_n \overset{D}{=} X \text{ , or } X_n \xrightarrow{D} X \text{ .}$$

Definition: A sequence of quantile functions $Q_n(\cdot)$ is said to converge in quantile to $Q(\cdot)$ if for every continuity point u of Q in $0 < u < 1$

$$\lim_{n\to\infty} Q_n(u) = Q(u) \text{ .}$$

Theorem 2A: Convergence in Distribution implies Convergence in Quantile, and conversely.

Proof. Let u be a continuity point of $Q(\cdot)$ $0 < u < 1$. Then one can choose a sequence ε_k converging to 0 such that, for each k, $Q(u)-\varepsilon_k$ and $Q(u) + \varepsilon_k$ are continuity points of $F(\cdot)$ and

$$F[Q(u) - \varepsilon_k] < u < F[Q(u) + \varepsilon_k] \text{ .}$$

Fix k. Convergence in distribution implies that one can choose N_k such that for all $n > N_k$

$$F_n(Q(u) - \varepsilon_k) < u < F_n(Q(u) + \varepsilon_k) \text{ .}$$

Consequently for $n \geq N_k$

$$Q(u) - \varepsilon_k \leq Q_n(u) \leq Q(u) + \varepsilon_k .$$

One can infer that $Q_n(u) \longrightarrow Q(u)$.

That convergence in quantile implies convergence in distribution follows by interchanging F and Q in the foregoing argument. A more probabilistic proof is the following. Let U be a standard uniform random variable, and define

$$\tilde{X}_n = Q_n(U) , \quad \tilde{X} = Q(U) .$$

which satisfy $\tilde{X}_n \overset{D}{=} X_n$, $\tilde{X} \overset{D}{=} X$. By hypothesis, $Q_n(u) \longrightarrow Q(u)$ almost surely Lebesgue measure on $(0,1)$, since at most a countable number of points are not continuity points of $Q(\cdot)$. Therefore $\tilde{X}_n \longrightarrow \tilde{X}$ almost surely, $\tilde{X}_n \xrightarrow{D} \tilde{X}$, and $F_n(x) \longrightarrow F(x)$ at all continuity points of $F(\cdot)$.

The foregoing argument is well known as the proof of a special case of the Skorohod Representation Theorem [Serfling (1981)].

<u>Theorem</u>. If $\lim_{n\to\infty} X_n \overset{D}{=} X$, one can choose random variables $\tilde{X}_n$, $\tilde{X}$ defined on a common probability space, such that

$$\tilde{X}_n \overset{D}{=} X_n , \ \tilde{X} \overset{D}{=} X , \ \lim_{n\to\infty} \tilde{X}_n = \tilde{X} \text{ almost surely.}$$

When $F_n(\cdot)$ converges to $F(\cdot)$, the moments of $F_n(\cdot)$ need not converge to $F(\cdot)$. Criteria for convergence of moments can be elegantly stated in terms of quantile functions.

<u>Definition</u>. For $r \geq 1$, define a distance between two distribution functions F_1 and F_2, with respective quantile functions Q_1 and Q_2 , by

$$d_r(F_1,F_2) = d_r(Q_1,Q_2) = \{\int_0^1 |Q_1(u) - Q_2(u)|^r du\}^{1/r}$$

This is an evaluation of the Vasershtein distance [(Major (1978)]

Define "convergence in r-mean quantile" of $Q_n(\cdot)$ to $Q(\cdot)$, denoted $Q_n \xrightarrow{r} Q$, by

$$\{d_r(Q_n,Q)\}^r = \int_0^1 |Q_n(u) - Q(u)|^r du \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

<u>Theorem 2B</u>. $Q_n \xrightarrow{r} Q$ if, and only if $\int_0^1 Q_n(u)\ ^r du \longrightarrow \int_0^1 |Q(u)|^r du$ and $Q_n(u) \longrightarrow Q(u)$ at continuity points of Q.

<u>Proof</u>. The "if" part of the theorem follows by integration theory, and in particular by Scheffé's theorem. The fact that $Q_n \xrightarrow{r} Q$ implies convergence of r-th moments also follows by integration theory. The following lemma seems novel and completes the proof.

<u>Lemma</u>. If $Q_n \xrightarrow{r} Q$ and $Q(u)$ and $Q_n(u)$ are non-decreasing functions of u, $0 \le u \le 1$, then $Q_n(u) \longrightarrow Q(u)$ at continuity points of Q.

<u>Proof</u>. We give a probabilist's proof. Let $Q(u)$ and $Q_n(u)$ be versions which are continuous from the left. Let U be standard uniform, and define $\tilde{X}_n = Q_n(U)$, $\tilde{X} = Q(U)$. One may argue that $E|\tilde{X}_n - \tilde{X}|^r \longrightarrow 0$; $\tilde{X}_n \xrightarrow{D} \tilde{X}$; $F_n(x) \longrightarrow F(x)$ at continuity points of $F(\cdot)$; $Q_n(u) \longrightarrow Q(u)$ at continuity points of Q.

It should be noted that

$$\int_{-\infty}^{\infty} |F_1(x) - F_2(x)|dx = \int_0^1 |Q_1(u) - Q_2(u)|du \le d_r(Q_1,Q_2) \text{ for } r \ge 1.$$

<u>Application to Sample Quantile Functions</u>. Let X be a random variable with distribution function $F(x)$ and quantile function $Q(u) = F^{-1}(u)$. Let $X_1, \ldots, X_n$ be a random sample

of X. The sample distribution function $\tilde{F}(x)$, $-\infty < x < \infty$, and the sample quantile function $Q(u)$, $0 \leq u \leq 1$ are defined by:

$$\tilde{F}(x) = \text{fraction of } X_1, \ldots, X_n \leq x ;$$

$$\tilde{Q}(u) = \tilde{F}^{-1}(u) = \inf \{x: \tilde{F}(x) \geq u\} .$$

It is well known that, as $n \longrightarrow \infty$,

$$\sup_{-\infty<x<\infty} |\tilde{F}(x) - F(x)| \longrightarrow 0 \text{ with probability one;}$$

if $E|X|^r < \infty$, $\int_{-\infty}^{\infty} |x|^r \, d\tilde{F}(x) \longrightarrow \int_{-\infty}^{\infty} |x|^r \, dF(x)$ with probability one .

Therefore, for $r \geq 1$, $E|X|^r < \infty$ implies

$$\int_0^1 |\tilde{Q}(u) - Q(u)|^r du \longrightarrow 0 \text{ with probability one.}$$

<u>Application to Extreme Value Distributions of Uniform Random Variables</u>. In section 5 the following representations of quantile functions will be used; it illustrates the use of quantile functions to demonstrate convergence in distribution.

Let $U_1, \ldots, U_n$ be standard uniform random variables. Let $Z = n \operatorname{Min}(U_1, \ldots, U_n)$. For any $x > 0$

$$1-F_Z(x) = \Pr[\min(U_1, \ldots, U_n) \geq \tfrac{x}{n}] = \{\Pr[U_1 \geq \tfrac{x}{n}]\}^n$$

$$= (1 - \tfrac{x}{n})^n .$$

One can solve for $x = Q_Z(u)$, and obtain:

$$Q_Z(u) = n\{1 - (1-u)^{1/n}\} .$$

For $0<y<1$, $y^{1/n} = e^{(\log y)/n} = 1 + \frac{1}{n} \log y + \theta \frac{1}{n^2} (\log y)^2 \frac{1}{y}$

where $0 < |\theta| < 1$. Consequently

$$Q_Z(u) = -\log(1-u) + \varepsilon_n(u)$$

where $\varepsilon_n(u) = \theta\frac{1}{n}|\log(1-u)|^2(1-u)^{-1} \longrightarrow 0$. Since $-\log(1-u)$ is the quantile function of a standard exponential random variable ξ, we conclude that $n \min(U_1,\ldots,U_n) \xrightarrow{D} \xi$.

For $Z = n\{\max(U_1,\ldots,U_n)-1\}$, one derives the quantile function

$$Q_Z(u) = n\{u^{1/n} - 1\} = -\log u^{-1} + \varepsilon_n(u)$$

where $\varepsilon_n(u) \longrightarrow 0$ as $n \longrightarrow \infty$, using the fact that

$$\Pr[\max(U_1,\ldots,U_n) \leq 1 - \frac{x}{n}] = \{\Pr[U_1 \leq 1 - \frac{x}{n}]\}^n = (1 - \frac{x}{n})^n$$

Since $-\log u^{-1}$ is the quantile function of $-\xi$ we conclude that $n\{\max(U_1,\ldots,U_n) - 1\} \xrightarrow{D} -\xi$.

3. Density Quantile Functions

A continuous random variable is traditionally described by its probability density function $f(x) = F'(x)$. In the quantile domain more insightful descriptions are given by the density-quantile function

$$fQ(u) = f(Q(u)) , \quad 0 \leq u \leq 1$$

and the quantile-density function

$$q(u) = Q'(u) , \quad 0 \leq u \leq 1 .$$

Differentiating the identity $FQ(u) = u$, one obtains

$$fQ(u)\, q(u) = 1 .$$

An important technique for computing $fQ(u)$ is as follows: (1) compute $Q(u)$; (2) differentiate to form $q(u)$; (3) take the reciprocal.

Another important function is the score function

$$J(u) = -(fQ)'(u) .$$

The Fisher score function is defined by

$$\psi(x) = -\frac{d}{dx} \log f(x) = -\frac{f'(x)}{f(x)} .$$

One may verify that

$$J(u) = \psi(Q(u)) .$$

Density-quantile functions have many important statistical applications. From the point of view of probability theory, we believe that their major insight is to provide definitions of the tail behavior of probability laws. An important characteristic of a distribution function $F(x)$ is the behavior of

1-F(x) and F(-x) as x tends to ∞, which we call its tail behavior. The study of extreme value distributions indicate that one can divide distributions into three main kinds of tail behavior:

(1) short tails (or limited type),

(2) medium tails (or exponential type),

(3) long tails (or Cauchy type).

<u>Definition</u>: A function L(u) is called <u>slowly varying</u> at u = 0 if for every y > 0

$$\frac{L(yu)}{L(u)} \longrightarrow 1 \text{ as } u \longrightarrow 0 .$$

An example of a slowly varying function is $L(u) = \{-\log u\}^{\beta}$.

<u>Definition</u>: A density quantile function is said to have a <u>left tail exponent</u> α_1 and a <u>right tail exponent</u> α_2 if, as $u \longrightarrow 0$,

$$\text{(1)} \quad \begin{aligned} L_1(u) &= u^{-\alpha_1} fQ(u) \text{ is slowly varying,} \\ L_2(u) &= u^{-\alpha_2} fQ(1-u) \text{ is slowly varying.} \end{aligned}$$

The exponents α_1 and α_2 can be any real numbers.

A function fQ(·) satisfying (1) is called regularly varying (see Seneta (1976)).

<u>Theorem</u> 3A: If tail exponents defined by (see Parzen (1979))

$$\text{(2)} \quad \begin{aligned} \alpha_1 &= \lim_{u \to 0} \frac{uJ(u)}{fQ(u)} \\ \alpha_2 &= \lim_{u \to 1} \frac{(1-u)J(u)}{fQ(u)} \end{aligned}$$

exist, then (1) holds. Formulas (2) provide a constructive, rather than a descriptive, definition of tail exponents; they apply when the density-quantile function is differentiable.

Proof: In the theory of regularly varying functions it is known that (2) implies (1) [see Seneta (1976), p. 7]; we briefly indicate the argument. Let

$$g(u) = -\frac{uJ(u)}{fQ(u)} - \alpha_1 .$$

Note that $g(t) \longrightarrow 0$ as $t \longrightarrow 0$.

Then

$$\frac{d}{du} \log fQ(u) = -\frac{J(u)}{fQ(u)} = \frac{g(u)}{u} + \frac{\alpha_1}{u} ,$$

$$\frac{d}{du} \log \{u^{-\alpha_1} fQ(u)\} = \frac{g(u)}{u} .$$

$$\log L_1(yu) - \log L_1(u) = \int_{yu}^{u} \frac{g(t)}{t}\, dt ,$$

$$\left|\log L_1(yu) - \log L_1(u)\right| \leq \{\max_{yu \leq t \leq u} |g(t)|\} |\log y| \longrightarrow 0.$$

In the study of the asymptotic distribution of sample maxima and minima, it is convenient to introduce a tail exponent

$$\gamma = \alpha - 1 .$$

We call α the <u>density-quantile tail exponent</u> and γ the <u>quantile tail exponent</u>. We define the following types of tail behavior:

$\gamma < 0, \ \alpha < 1$	short tails (or limited type)
$\gamma = 0, \ \alpha = 1$	medium tails (or exponential type)
$\gamma > 0, \ \alpha > 1$	long tails (or Cauchy type)
$\gamma < -1, \alpha < 0$	super-short tails .

The basic existence theorem of extreme value theory, due to Frechet (1927), Fisher and Tippett (1928), and Gnedenko (1947) can be expressed as follows.

Theorem 3B. The distributions that can arise as the asymptotic distribution of an extreme value are, up to a location and scale parameter, the distributions of the following functions of a standard exponential random variable, ξ, which have the quantile functions listed.

Tail Behavior	Maximum	Minimum
Medium Tail $\gamma = 0, \ \alpha = 1$	$-\log \xi$ $-\log \log u^{-1}$ Gumbel distribution	$\log \xi$ $\log \log (1-u)^{-1}$
Short Tail $\gamma < 0, \ \alpha < 1$	$-\xi^{-\gamma}$ $-\{\log u^{-1}\}^{-\gamma}$	ξ^{γ} $\{\log (1-u)^{-1}\}^{-\gamma}$ Weibull distribution for $-1 \leq \gamma < 0$
Long Tail $\gamma > 0, \ \alpha > 1$	$\xi^{-\gamma}$ $\{\log u^{-1}\}^{-\gamma}$	$-\xi^{-\gamma}$ $-\{\log (1-u)^{-1}\}^{-\gamma}$

The density quantile functions and right and left density-quantile tail exponents are listed in the following table:

Tail Behavior	Maximum	Minimum
Medium Tail $\gamma = 0$	$u \log u^{-1}$ Left $\alpha=1$, Right $\alpha=1$	$(1-u) \log (1-u)^{-1}$ Left $\alpha=1$, Right $\alpha=1$
Short Tail $\gamma < 0$	$-\frac{1}{\gamma} u\{\log u^{-1}\}^{-(1+\gamma)}$ Left $\alpha=1$, Right $\alpha=1+\gamma$	$-\frac{1}{\gamma} (1-u)\{\log (1-u)^{-1}\}^{-(1+\gamma)}$ Left $\alpha=1+\gamma$, Right $\alpha=1$
Long Tail $\gamma > 0$	$\frac{1}{\gamma} u\{\log u^{-1}\}^{-(1+\gamma)}$ Left $\alpha=1$, Right $\alpha=1+\gamma$	$\frac{1}{\gamma} (1-u)\{\log (1-u)^{-1}\}^{-(1+\gamma)}$ Left $\alpha=1+\gamma$, Right $\alpha=1$

Characterization of moments that exist for given tail exponents. When $\gamma_1 > 0$, the quantile function $Q(u)$ is at $u = 0$, a regularly varying function with index γ_1, in the sense that $Q(u) = u^{-\gamma_1}L(u)$ where $L(u)$ is slowly varying: when $\gamma_2 > 0$ the quantile function is, at $u = 1$, a regularly varying function with index γ_2 in the sense that

$$Q(1-u) = u^{-\gamma_2}L(u) .$$

The integral of a regularly varying function of index γ has the same convergence properties as $u^{-\gamma}$. Therefore the k-th absolute moment

$$\int_0^1 |Q(u)|^k du < \infty \text{ iff } k < 1/\gamma_1 \text{ and } k < 1/\gamma_2 .$$

When $\gamma_1 \le 0$ and $\gamma_2 \le 0$, then all moments are finite.

Similarly one may prove the following theorem mentioned by Stigler (1974).

Theorem 3C. Let $Q(u)$ be regularly varying with positive left and right tail exponents. Let k and δ be constants so that $k\delta = 1$. Then $\int_0^1 |Q(u)|^k du < \infty$ if and only if

$$\int_0^1 \{u(1-u)\}^\delta \, dQ(u) = \int_{-\infty}^{\infty} \{F(x)(1-F(x))\}^\delta dx < \infty \quad .$$

4. Extreme Value Distribution Theory

In this section we summarize some of the basic results of the theory of extreme values, and show how increased insight into their proofs and application is obtained by thinking in terms of quantile functions, and density-quantile functions.

Let $X_1,\ldots,X_n$ be a random sample of a random variable X, and let

$$\text{Max}(n) = \text{Max}(X_1,\ldots,X_n) \ , \quad \text{Min}(n) = \text{Min}(X_1,\ldots,X_n) \ .$$

The aim is to determine if

$$(1) \quad F_{\text{Max}(n)}\,(a_n+b_n x) = \Pr\left[\frac{\text{Max}(n)-a_n}{b_n} \le x\right] \xrightarrow{D} G(x)$$

$$(2) \quad F_{\text{Min}(n)}\,(c_n+d_n x) = \Pr\left[\frac{\text{Min}(n)-c_n}{d_n} \le x\right] \xrightarrow{D} H(x)$$

holds for suitable constants a_n, b_n, c_n, d_n and distribution functions G(x) and H(x), all of which are to be determined. In terms of convergence of quantile functions, (1) and (2) are equivalent to

$$(3) \quad \frac{1}{b_n}\{Q\,[\text{Max}(n);u] - a_n\} \xrightarrow{D} G^{-1}(u) \ ,$$

$$(4) \quad \frac{1}{d_n}\{Q[\text{Min}(n);u] - c_n\} \xrightarrow{D} H^{-1}(u) \ ,$$

where hereafter we write *Q(X;u) to denote the quantile function* of a random variable X.

We state theorems which summarize the theory of extreme value distributions in a table. Theorem 4A states the normalizing constants in terms of a general quantile function Q. Theorem 4B provides more precise formulas for normalizing

constants, assuming certain asymptotic expressions for $Q(u)$. We then outline the extreme value distributions of some frequently encountered probability distributions.

The reader should verify the following important consequence of Theorem 4A: $\text{Max}(X_1,\ldots,X_n)$ has the same right tail exponent as X and $\text{Min}(X_1,\ldots,X_n)$ has the same left tail exponent as X.

<u>Theorem</u> <u>4A</u>. Let α be the right tail exponent (or left tail exponent) of the density-quantile function $fQ(u)$ of a random variable X with quantile function $Q(u)$. Let $\gamma = \alpha - 1$. Then the maximum (minimum) has the following asymptotic distribution (where ξ denotes a standard exponential random variable):

Density Quantile Tail Exponent	Maximum	Minimum
Medium Tail $\alpha = 1$ $\gamma = 0$	$\dfrac{\text{Max}(n)-Q(1-\frac{1}{n})}{Q(1-\frac{1}{ne})-Q(1-\frac{1}{n})} \xrightarrow{D} -\log\xi$	$\dfrac{\text{Min}(n)-Q(\frac{1}{n})}{Q(\frac{1}{ne})-Q(\frac{1}{n})} \xrightarrow{D} \log\xi$
Short Tail $\alpha < 1$ $\gamma < 0$ Weibull $-1<\gamma<0$	$\dfrac{\text{Max}(n)-Q(1)}{Q(1)-Q(1-\frac{1}{n})} \xrightarrow{D} -\xi^{-\gamma}$	$\dfrac{\text{Min}(n)-Q(0)}{Q(\frac{1}{n})-Q(0)} \xrightarrow{D} \xi^{-\gamma}$
Long Tail $\alpha > 1$ $\gamma > 0$	$\dfrac{\text{Max}(n)}{Q(1-\frac{1}{n})} \xrightarrow{D} \xi^{-\gamma}$	$\dfrac{\text{Min}(n)}{-Q(\frac{1}{n})} \xrightarrow{D} -\xi^{-\gamma}$

<u>Proof</u>. One verifies that the conditions on $fQ(u)$ in the foregoing table imply the following conditions on the quantile function $Q(u)$, which are shown in section 5 to be necessary and sufficient conditions for the above asymptotic distributions to hold:

Tail Exponent	Maximum : for every y as $n \longrightarrow \infty$	Minimum : for every y, as $n \longrightarrow \infty$.
Medium Tail $\gamma = 0$	$\dfrac{Q(1-\frac{y}{n})-Q(1-\frac{1}{n})}{Q(1-\frac{1}{ne})-Q(1-\frac{1}{n})} \longrightarrow \log y$	$\dfrac{Q(\frac{y}{n})-Q(\frac{1}{n})}{Q(\frac{1}{ne})-Q(\frac{1}{n})} \longrightarrow \log y$
Short Tail $\gamma < 0$	$\dfrac{Q(1-\frac{y}{n})-Q(1)}{Q(1-\frac{1}{n})-Q(1)} \longrightarrow y^{-\gamma}$	$\dfrac{Q(\frac{y}{n})-Q(0)}{Q(\frac{1}{n})-Q(0)} \longrightarrow y^{-\gamma}$
Long Tail $\gamma > 0$	$\dfrac{Q(1-\frac{y}{n})}{Q(1-\frac{1}{n})} \longrightarrow y^{-\gamma}$	$\dfrac{Q(\frac{y}{n})}{Q(\frac{1}{n})} \longrightarrow y^{-\gamma}$

<u>Theorem</u> <u>4B</u>. Sufficient conditions for asymptotic distributions are that the quantile functions have the following representations (where $B > 0$):

	Maximum As $u \longrightarrow 1$	Minimum As $u \longrightarrow 0$
Medium Tail $\gamma = 0$	$Q(u) \sim A+B\{\log (1-u)^{-1}\}^{\beta}$	$Q(u) \sim A-B\{\log u^{-1}\}^{\beta}$
Short Tail $\gamma < 0$	$Q(1)-Q(u) \sim B\ (1-u)^{-\gamma}$	$Q(u) - Q(0) \sim B\ u^{-\gamma}$
Long Tail $\gamma > 0$	$Q(u) \sim A+B\ (1-u)^{-\gamma}$	$Q(u) \sim A - B\ u^{-\gamma}$

Under these sufficient conditions, the extreme value asymptotic distributions are as follows:

Medium Tail $\gamma = 0$	$\frac{\text{Max}(n)-B\{\log n\}^{\beta}}{B\beta(\log n)^{\beta-1}} \xrightarrow{D} -\log \xi$	$\frac{\text{Min}(n)+B\{\log n\}^{\beta}}{B\beta(\log n)^{\beta-1}} \xrightarrow{D} \log \xi$
Short Tail $\gamma < 0$	$\frac{1}{B}n^{-\gamma}\{\text{Max}(n)-Q(1)\} \xrightarrow{D} -\xi^{-\gamma}$	$\frac{1}{B}n^{-\gamma}\{\text{Min}(n)-Q(0)\} \xrightarrow{D} \xi^{-\gamma}$
Long Tail $\gamma > 0$	$\frac{\text{Max}(n)}{Bn^{\gamma}} \xrightarrow{D} \xi^{-\gamma}$	$\frac{\text{Min}(n)}{Bn^{\gamma}} \xrightarrow{D} -\xi^{-\gamma}$

It should be noted that for medium tail distributions whose quantile function has the special representation given in Theorem 4B the extreme value distribution depends on the parameter β which we call a <u>shape</u> parameter. The scale divisor b_n such that $(\text{Max}(n) - a_n)/b_n$ tends to a limiting distribution has the following behavior:

Scale Divisor b_n	Tail Exponent γ and Shape Parameter β
Constant	$\gamma = 0$, $\beta = 1$
Tends to 0	$\gamma < 0$ or $\gamma = 0$, $\beta < 1$
Tends to ∞	$\gamma > 0$ or $\gamma = 0$, $\beta > 1$

To state the asymptotic distribution of extreme values of familiar standard probability laws let us introduce two important constants a_n^* and b_n^* which occur in the tail representation of $\Phi^{-1}(u)$.

<u>Lemma</u>: $\Phi^{-1}(1-\frac{y}{n}) = a_n^* - b_n^* \log y$, where

$$a_n^* = (2\log n)^{\frac{1}{2}} - \frac{1}{2}(2\log n)^{-\frac{1}{2}}(\log\log n + 4\pi)$$

$$b_n^* = (2\log n)^{-\frac{1}{2}} .$$

<u>Proof</u>: Let $u_n = \Phi^{-1}(1-\frac{x}{n})$. Then $\Phi\ (x) \sim (1/x)\phi(x)$ implies

$$u_n^2 = 2\log n - \log 2\pi - 2\log u_n - 2\log x$$

$$\log u_n = \frac{1}{2}\log 2 + \frac{1}{2}\log\log n + o(1) .$$

Probability Distribution and Quantile Function	Maximum Right Tail Exponent	Minimum Left Tail Exponent
Normal $\Phi^{-1}(u)$	$(2\log n)^{\frac{1}{2}}\{\mathrm{Max}(n)-a_n^*\}$ $\xrightarrow{D} -\log\xi$ $\alpha = 1$, $\beta = 0.5$	$(2\log n)^{\frac{1}{2}}\{\mathrm{Min}(n)+a_n^*\}$ $\xrightarrow{D} \log\xi$, $\alpha = 1$, $\beta = 0.5$
Exponential $\log(1-u)^{-1}$	$\mathrm{Max}(n) - \log n$ $\xrightarrow{D} -\log\xi$ $\alpha = 1$, $\beta = 1$	$n\,\mathrm{Min}(n)$ $\xrightarrow{D} \xi$, $\alpha = 0$
Uniform u	$n\{\mathrm{Max}(n) - 1\}$ $\xrightarrow{D} -\xi$ $\alpha = 0$	$n\,\mathrm{Min}(n)$ $\xrightarrow{D} \xi$, $\alpha = 0$

Weibull $\{\log(1-u)^{-1}\}^{\beta}$ $0<\beta<1$	$\beta(\log n)^{1-\beta}\{\mathrm{Max}(n)-(\log n)^{\beta}\}$ $\longrightarrow -\log\xi$ $\alpha = 1\ ,\ \beta = \beta$	$n^{\beta}\mathrm{Min}(n)$ $\longrightarrow \xi^{\beta}$ $\alpha = \beta$
Cauchy $Q(u) = \tan\pi(u-\frac{1}{2})$ $= -\frac{\cos \pi u}{\sin \pi u}$ $\sim -\frac{1}{\pi u}$ as $u \longrightarrow 0$	$\pi\ \frac{\mathrm{Max}(n)}{n}$ $\longrightarrow \xi^{-1}$ $\alpha = 2$	$\pi\ \frac{\mathrm{Min}(n)}{n}$ $\longrightarrow \xi^{-1}$ $\alpha = 2$

Finally, we give an example which illustrates how one uses the quantile approach to identify the asymptotic extreme value distributions of a continuous distribution function $F(x)$. The procedure is: (1) compute $Q(u)$, $fQ(u)$, $J(u)$; (2) compute left and right tail exponents α_1 and α_2 which identifies the type of extreme value distributions; (3) compute $Q(1/n)$ and $Q(1-(1/n))$ which determines the norming constants.

Example: One-sided stable distribution of index 1/2 has distribution function

$$F(x) = 2\ [1 - \Phi(x^{-\frac{1}{2}})]\ .$$

One obtains $Q(u)$ by solving $u = F(x)$ for $x = Q(u)$:

$$Q(u) = \{\Phi^{-1}(1-\tfrac{u}{2})\}^{-2}$$

$$fQ(u) = \phi\Phi^{-1}(1-\tfrac{u}{2})\{\Phi^{-1}(1-\tfrac{u}{2})\}^{3}$$

$$J(u) = \tfrac{3}{2}\ \{\Phi^{-1}(1-\tfrac{u}{2})\}^{2} - \tfrac{1}{2}\{\Phi^{-1}(1-\tfrac{u}{2})\}^{4}\ .$$

The tail exponents are computed to be

$$\alpha_1 = 1 \text{(medium tails)}, \quad \alpha_2 = 3 \text{(long tails)} .$$

One infers that

$$\frac{\text{Min}(n) - Q(1/n)}{Q(1/ne) - Q(1/n)} \xrightarrow{D} \log \xi ,$$

$$\frac{\text{Max}(n)}{Q(1-(1/n))} \xrightarrow{D} \xi^{-2} .$$

To compute $Q(1 - (1/n))$, define $q(u) = \{\phi\Phi^{-1}(u)\}^{-1}$. Then for u near 1

$$\Phi^{-1}(1-\tfrac{u}{2}) = \int_{0.5}^{1-0.5u} q(u)du \doteq \tfrac{1}{2}(1-u)q(0) ,$$

$$Q(u) \doteq \{2\phi\Phi^{-1}(0)\}^2 (1-u)^{-2} = \frac{2}{\pi} (1-u)^{-2}$$

We conclude that

$$\text{Max}(n)/ \frac{2}{\pi} n^2 \xrightarrow{D} \xi^{-2} .$$

To find the norming constants of Min(n), write

$$Q(\tfrac{y}{n}) = \{\Phi^{-1}(1 - \tfrac{y}{2n})\}^{-2} = \{a_n^* - b_n^* \log (y/2)\}^{-2}$$

$$Q(1/ne) - Q(1/n) \sim \frac{2b_n^*}{a_n^{*3}} \sim 2(\log n)^{-2}$$

We conclude that

$$\tfrac{1}{2}(\log n)^2 \{\text{Min}(n) - Q(1/n)\} \xrightarrow{D} \log \xi.$$

Similarly one may find the extreme value distributions of the lognormal distribution with quantile function $Q(u) = \exp \Phi^{-1}(u)$.

5. Quantile Derivation of Extreme Value Distributions.

Let $X_1,\ldots,X_n$ be a random sample of a random variable X with quantile function Q(u). In this section we outline a derivation of the distribution of $\max(X_1,\ldots,X_n)$ and $\min(X_1,\ldots,X_n)$ by quantile methods.

Tail exponents of density-quantile functions have been introduced as criteria that can be easily applied in practice to determine the asymptotic extreme value distributions of continuous random variables. They are only sufficient conditions. The following conditions expressed in terms of quantile functions are necessary and sufficient conditions for convergence, and apply to arbitrary distributions.

Definition: A quantile function Q(u) , $0 \le u \le 1$, belongs to the class described below if it satisfies the condtion given for any $y > 0$ and any sequence y_n tending to y as n tends to ∞ .

Left Short Tail, with left quantile tail exponent $\gamma < 0$, if Q(0) is finite and

$$\lim_{n\to\infty} \frac{Q(\frac{y_n}{n})-Q(0)}{Q(\frac{1}{n})-Q(0)} = y^{-\gamma}$$

or equivalently $u^{\gamma}\{Q(u) - Q(0)\}$ is slowly varying at $u = 0$.

Right Short Tail, with right quantile tail exponent $\gamma < 0$, ff Q(1) is finite and

$$\lim_{n\to\infty} \frac{Q(1-\frac{y_n}{n})-Q(1)}{Q(1-\frac{1}{n})-Q(1)} = y^{-\gamma} ,$$

or equivalently $(1-u)^{\gamma}\{Q(u) - Q(1)\}$ is slowly varying at $u = 1$.

<u>Left Long Tail</u>, with left quantile tail exponent $\gamma > 0$ if $Q(0) = -\infty$ and

$$\lim_{n\to\infty} \frac{Q(\frac{y}{n})}{Q(\frac{1}{n})} = y^{-\gamma}$$

or equivalently $u^{\gamma}Q(u)$ is slowly varying at $u = 0$.

<u>Right Long Tail</u>, with right quantile tail exponent $\gamma > 0$ if $Q(1) = \infty$ and

$$\lim_{n\to\infty} \frac{Q(1-\frac{y}{n})}{Q(1-\frac{1}{n})} = y^{-\gamma}$$

or equivalently $(1-u)^{\gamma}Q(u)$ is slowly varying at $u = 1$.

<u>Medium Tail</u>, with quantile tail exponent $\gamma = 0$, if

$$\text{Right tail} \lim_{u\to 1} \frac{Q(1-\frac{y}{n}) - Q(1-\frac{1}{n})}{Q(1-\frac{1}{ne}) - Q(1-\frac{1}{n})} = \log y$$

$$\text{Left tail} \lim_{u\to 0} \frac{Q(\frac{y}{n}) - Q(\frac{1}{n})}{Q(\frac{1}{ne}) - Q(\frac{1}{n})} = \log y \ .$$

By representing $X_i = Q(U_i)$ where $U_1,\ldots,U_n$ are standard uniform variables we have the basic representation:

$$Q[\max(X_1,\ldots,X_n);\ u] = Q(Q[\max(U_1,\ldots,U_n);u])$$

$$Q[\min(X_1,\ldots,X_n);\ u] = Q(Q[\min(U_1,\ldots,U_n);u])$$

To write an expression for the quantile function of extreme values of uniform random variables, let ξ be a standard exponential random variable, and define

$$R_0(u) = \log(1-u)^{-1} = Q[\xi;u],\ R_1(u) = \log u^{-1} = -Q[-\xi;u]$$

Of use later will be the fact that $R_1(u) = R_0(1-u)$. At the end of section 3 it is shown that at each u in $0 < u < 1$, asympotically, as $n \longrightarrow \infty$,

$$Q[\min(U_1,\ldots,U_n);u] = \frac{1}{n} \tilde{R}_0(u) \ , \ \tilde{R}_0(u) = R_0(u) + \varepsilon_n(u)$$

$$Q[\max(U_1,\ldots,U_n);u] = 1 - \frac{1}{n} \tilde{R}_1(u) \ , \ \tilde{R}_1(u) = R_1(u) + \varepsilon_n(u)$$

where $\varepsilon_n(u)$ denotes a remainder tending to 0 as $n \longrightarrow \infty$

The asymptotic distributions of extreme values stated in section 4 is proved using quantile functions by the following argument.

<u>Long tailed distribution with quantile tail exponent</u> $\gamma > 0$.

$$Q\left[\frac{\min(X_1,\ldots,X_n)}{-Q(1/n)};u\right] = \frac{Q(\tilde{R}_0(u)/n)}{-Q(1/n)} \longrightarrow -\{R_0(u)\}^{-\gamma}$$

$$Q\left[\frac{\max(X_1,\ldots,X_n)}{Q(1-1/n))};u\right] = \frac{Q(1-(1/n)\tilde{R}_1(u))}{Q(1-(1/n))} \longrightarrow \{R_1(u)\}^{-\gamma} .$$

<u>Short tailed distribution with quantile tail exponent</u> $\gamma < 0$.

$$Q\left[\frac{\min(X_1,\ldots,X_n)-Q(0)}{Q(1/n)-Q(0)};u\right] = \frac{Q[\tilde{R}_0(u)/n]-Q(0)}{Q(1/n)-Q(0)} \longrightarrow \{R_0(u)\}^{-\gamma}$$

$$Q\left[\frac{\max(X_1,\ldots,X_n)-Q(1)}{Q(1)-Q(1-(1/n))};u\right] = -\frac{Q(1-(1/n)\tilde{R}_1(u))-Q(1)}{Q(1-(1/n))-Q(1)} \longrightarrow -\{R_1(u)\}^{-\gamma} .$$

<u>Medium tail distribution</u> ($\gamma = 0$)

$$Q\left[\frac{\min(X_1,\ldots,X_n)-Q(1/n)}{Q(1/ne)-Q(1/n)};u\right] = -\frac{Q(\tilde{R}_0(u)/n)-Q(1/n)}{Q(1/ne)-Q(1/n)} \longrightarrow \log R_0(u)$$

$$Q\left[\frac{\max(X_1,\ldots,X_n)-Q(1-(1/n))}{Q(1-(1/ne))-Q(1-(1/n))};u\right] = \frac{Q(1-\tilde{R}_1(u)/n)-Q(1-(1/n))}{Q(1-(1/ne))-Q(1-(1/n))}$$

$$\longrightarrow -\log R_1(u) .$$

<u>Faster Convergence</u>. To speed up the rate of convergence to the asymptotic distribution one can consider powers of extreme values, using the following lemma. [Compare Weinstein (1973)]

<u>Lemma</u>. Let M_n be a sequence of random variables, a_n, $b_n > 0$ sequences of constants, and Z a random variable such that

$$Z_n = \frac{M_n - a_n}{b_n} \xrightarrow{D} Z ,$$

$$b_n/a_n \longrightarrow 0 .$$

Then for any $k > 0$

$$Z_n^{(k)} = \frac{M_n^k - a_n^k}{k b_n a_n^{k-1}} \xrightarrow{D} Z$$

<u>Proof</u>: Verify that

$$\begin{aligned} Q[Z_n^{(k)};u] &= \{k b_n a_n^{k-1}\}^{-1}\{Q[M_n^k;u] - a_n^k\} \\ &= \{k b_n/a_n\}^{-1}\{(1+\frac{b_n}{a_n} Q[Z_n;u])^k - 1\} \\ &= Q[Z_n;u] + 0(\frac{b_n}{a_n}) \longrightarrow Q[Z;u] . \end{aligned}$$

The criteria stated in Section 4 as sufficient conditions for extreme value distributions of absolutely continuous distribution include quantile domain versions of the following classic sufficient conditions [see Galambos (1979), p. 93]:

$$\text{(1)} \qquad \lim_{x} \frac{xf(x)}{1-F(x)} = \frac{1}{\gamma}$$

$$\text{(2)} \qquad \lim_{x} \frac{d}{dx}\left[\frac{1-F(x)}{f(x)}\right] = \lim_{x} \left\{1 + \frac{(1-F(x)f'(x)}{f^2(x)}\right\} = 0$$

Letting $x = Q(u)$, and $q(u) = 1/fQ(u)$, one can rewrite (1) and (2)

$$\text{(3)} \qquad \lim_{u \to 1} \frac{(1-u)q(u)}{Q(u)} = \gamma$$

$$\text{(4)} \qquad \lim_{u \to 1} \frac{(1-u)J(u)}{fQ(u)} = 1$$

(3) is a sufficient condition for $Q(u)$ to be regularly varying with index γ, while (4) is a sufficient condition for $fQ(u)$ to be regularly varying of index 1.

6. Acknowledgement.

I would like to express my warm appreciation to my colleague Professor Bill Parr for his help in the form of literature references and stimulating conversation.

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